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A note on hyperharmonic and polyharmonic functions

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Abstract

We show that the spaces of harmonic functions with respect to the Poincaré metric in the unit ball B^N in $\mathbb{R}^N$ have many different properties depending upon whether N is even or odd.

Keywords: Hyperbolically harmonic functions; Polyharmonic functions; Hypergeometric functions

1. Introduction

For a C^2 function u on a domain G in $\mathbb{R}^N$ $(N \geqslant 2)$, let

$$\Delta_h u(x) = \left(1 - |x|^2\right)\left[\left(1 - |x|^2\right)\Delta u(x) + 2(N-2)Ru(x)\right],$$

where $\Delta u = \sum_{j=1}^{N}(\partial^2 u/\partial x_j^2)$ and $Ru = \sum_{j=1}^{N} x_j(\partial u/\partial x_j)$.

We put $h(G) = \{u \in C^2(G)\colon\ \Delta_h u = 0\}$. We say u is hyperharmonic in G if $u \in h(G)$.

It is well known that the uniform limit of a sequence of hyperharmonic functions in the Poincaré ball $B^N = \{x \in \mathbb{R}^N\colon\ |x| < 1\}$ is hyperharmonic. In this paper we show that any function that is hyperharmonic and real analytic on a domain $G \subset \mathbb{R}^{2N}$ is N-polyharmonic. Consequently, the uniform limit of a sequence of hyperharmonic and real analytic functions on G is hyperharmonic and real analytic on G.

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On the other hand, if N is odd and $0 \in G \subset \mathbb{R}^N$, then there are no functions on G, except constants, that are hyperharmonic, real analytic and polyharmonic of any order $k \geqslant 1$.

There is another reason for which the case $N = 2n$ is interesting. Namely, there is an infinite-dimensional class of hyperharmonic polynomials on B^{2n}, and moreover, this class is dense in $h(B^{2n})$, the class of hyperharmonic functions on B^{2n}, in the topology of uniform convergence on compact sets in B^{2n}. On the other hand, $h(B^{2n+1})$ does not contain any non-constant polynomial and, moreover, if $u \in h(B^{2n+1}) \cap C^{2n}(\overline{B^{2n+1}})$ then u is a constant. In fact, we show that a much weaker condition than this implies that a hyperharmonic function on B^{2n+1} must be constant.

We show that functions in the hyperharmonic Hardy space $h^p(B^N)$ have boundary value in the sense of distributions.

2. Polyharmonic and hyperharmonic functions

For a domain $G \subset \mathbb{R}^N$ let $H_m(G)$, $m \geqslant 1$, be the class of functions polyharmonic of degree m (m-polyharmonic) in G, i.e., solutions of the equation $\Delta^m u = 0$, where Δ^m is the Euclidean Laplacian iterated m times (cf. [3]). In particular, $H(G) = H_1(G)$ is the class of functions harmonic in G. It is known that $H_m(G) \subset A(G)$, where $A(G)$ is the class of real analytic functions on G (see [3]).

We note that if $G \cap S^{N-1}$ ($S^{N-1} = \partial B^N$) is not empty, then there are solutions of the equation $\Delta_h u = 0$ which are not real analytic in G.

For example, if $N = 4$, then the function u defined by

$$u(x) = \begin{cases} |x|^2 - |x|^{-2} - 4\log|x|, & |x| \geqslant 1, \\ 0, & |x| \leqslant 1, \end{cases}$$

is C^2 on $\mathbb{R}^4$, and $\Delta_h u = 0$ for all x; clearly u is not real analytic.

It is clear that $h(G) = H(G)$ if $N = 2$. If $N \geqslant 3$ and G contains the origin, then $h(G) \cap H(G) = \{\text{constants}\}$.

Theorem 2.1. *Let G be a domain in $\mathbb{R}^N$, $N \geqslant 3$, and $u \in h(G) \cap A(G)$.*

(i) *If N is even, then $u \in H_{N/2}(G)$. If in addition $u \neq \text{const}$ and $0 \in G$, then $u \in H_{N/2}(G) \setminus H_{N/2-1}(G)$.*
(ii) *If N is odd, $0 \in G$ and $u \in H_m(G)$ for some m, then u is constant.*

Theorem 2.1 is easily proved by successive applications of the following two lemmas.

Lemma 2.2. *Let $u \in A(G)$, where G is a domain in $\mathbb{R}^N$ containing the origin. If $R_s u := su + Ru = 0$, for some $s > 0$, then $u = 0$. If $Ru = 0$, then $u = \text{const}$.*

Proof. This follows from the formula $R_s u = |x|^{-s} R(|x|^s u)$ and the uniqueness theorem. □

Lemma 2.3. *If $u \in h(G) \cap A(G)$, where G is a domain in $\mathbb{R}^N$, then*

$$\left(1-|x|^2\right)\Delta^m u = 2(2m-N)R_{m-1}\Delta^{m-1}u, \quad m \geqslant 1. \tag{2.1}$$

(If $m=1$, then $\Delta^{m-1}u=u$.)

Proof. If $m=1$, then (2.1) follows from the equation $\Delta_h u=0$. On the other hand, by differentiation and using the formula $\Delta R_s = R_{s+2}\Delta$, we deduce from (2.1) that

$$\left(1-|x|^2\right)\Delta^{m+1}u - 4R\Delta^m u - 2N\Delta^m u = 2(2m-N)R_{m+1}\Delta^m u.$$

This implies, after a little work,

$$\left(1-|x|^2\right)\Delta^{m+1}u = 2(2m+2-N)R_m\Delta^m u.$$

Now the lemma is proved by induction on m. □

Corollary 2.4. *Let N be even and $G \subset \mathbb{R}^N$. Then the uniform limit of a sequence of hyperharmonic and real analytic functions in G is hyperharmonic and real analytic in G.*

This is easily deduced from Theorem 2.1(i) and the analogous fact for polyharmonic functions.

Corollary 2.4 is interesting only when $G \cap S^{N-1}$ is not empty, since otherwise the operator Δ_h is elliptic in G.

Note the following consequence of the proof of Lemma 2.3.

Corollary 2.5. *If $u \in C^{2n}(G)$, $G \subset \mathbb{R}^{2n}$, and $\Delta_h u = 0$ in G, then $u \in A(G)$.*

3. Hyperharmonic functions having a distribution value

In [7] it is shown that if $f \in h(B^N)$ then there exists a unique sequence of harmonic homogeneous polynomials f_k, of degree k, $f_k \in \mathcal{H}_k(\mathbb{R}^N)$, such that

$$f(x) = \sum_{k=0}^{\infty} F_k(x) f_k(x), \quad x \in B^N,$$

where $F_k(x) = F(k, 1-N/2, k+N/2; |x|^2)$, $k \geqslant 0$ (as usual $F(a,b,c;\cdot)$ denotes the hypergeometric function with parameters a, b, c (see [6, Chapter II])). Note that if N is even, then F_k, $k \geqslant 1$, is a polynomial of degree $N-2$, while if N is odd, F_k is only of class C^{N-2} on $\overline{B^N}$. More precisely the following theorem was proven.

Theorem 3.1. *If u is a hyperharmonic function in B^N, then there exists a unique sequence of harmonic homogeneous polynomials $f_k \in \mathcal{H}_k(\mathbb{R}^N)$ such that*

$$u(x) = \sum_{k=0}^{\infty} F_k(x) f_k(x), \quad x \in B^N, \tag{3.1}$$

the series converging uniformly and absolutely on compact subsets of B^N. Conversely, the sum of any such series that converges uniformly on compact subsets of B^N is hyperharmonic in B^N.

As a corollary of Theorem 3.1 we have that the class of hyperharmonic polynomials is dense in $h(B^{2n})$. The situation is completely different when N is odd. Since every polynomial is a polyharmonic function, it follows from Theorem 2.1(ii) that there exists no non-constant hyperharmonic polynomial on $\mathbb{R}^N$ when N is odd. Our next theorem shows that a more general fact is true.

Theorem 3.2. *Let $u \in h(B^{2n+1})$. If*

$$\int_{S^{2n}} R^{2n}u(r\xi)\phi(\xi)\,d\sigma(\xi) = o\left(\log\frac{1}{1-r}\right), \quad r \to 1, \tag{3.2}$$

for every $\phi \in C^\infty(S^{2n})$, then u is constant.

Here, as usual, σ denotes the normalized rotation invariant measure on S^{2n}.

Proof. From (3.1) and the orthogonality in $L^2(S^{2n}, d\sigma)$ of the sequence $f_k(y)$ we get

$$F_k(r)r^k \int_{S^{2n}} f_k(y)^2\,d\sigma(y) = \int_{S^{2n}} u(ry)f_k(y)\,d\sigma(y), \tag{3.3}$$

where we write $F_k(r) = F_k(ry)$, $y \in S^{2n}$.

It is well known that

$$F^{(j)}\left(k, 1 - \frac{2n+1}{2}, k + \frac{2n+1}{2}; r^2\right), \quad k \geqslant 1,$$

is bounded for $j = 0, 1, 2, \ldots, 2n-1$, and

$$F^{(2n)}\left(k, 1 - \frac{2n+1}{2}, k + \frac{2n+1}{2}; r^2\right) \sim C\log\frac{1}{1-r}, \quad r \to 1.$$

If $k \geqslant 1$, we find from (3.2) and (3.3) that

$$\left(r\frac{d}{dr}\right)^{(2n)}\left(F_k(r)r^k \int_{S^{2n}} f_k(y)^2\,d\sigma(y)\right)$$
$$= \int_{S^{2n}} R^{2n}u(ry)f_k(y)\,d\sigma(y) = o\left(\log\frac{1}{1-r}\right).$$

Hence, $f_k = 0$, for $k = 1, 2, \ldots$, i.e., $u = \text{const}$. □

In particular, the following is true.

Corollary 3.3. *If $u \in h(B^{2n+1}) \cap C^{2n}(\overline{B^{2n+1}})$, then u is constant.*

We say that a function u defined on B^N has a distribution value on S^{N-1} if

$$\lim_{r\to 1} \int_{S^{N-1}} u(ry)\phi(y)\,d\sigma(y)$$

exists for each test function $\phi \in C^\infty(S^{N-1})$.

Recall that the tangential derivatives of $u \in C^1(B^N)$ are defined by

$$T_{i,j}u(x) = x_i \frac{\partial u}{\partial x_j}(x) - x_j \frac{\partial u}{\partial x_i}(x), \quad 1 \leqslant i, j \leqslant N.$$

The following result is closely related to Theorem 3.2.

Theorem 3.4. *If u is a hyperharmonic function on B^N having a distribution boundary value and $X = R^k Y$, with Y tangential, then Xu has a distribution value when $k \leqslant N-2$. If $k = N-1$, then*

$$\int_{S^{N-1}} R^{N-1} Y u(ry)\phi(y)\, d\sigma(y) = O\left(\log \frac{1}{1-r}\right)$$

for each $\phi \in C^\infty(S^{N-1})$.

Proof. By direct calculation, since $(1-|x|^2)\Delta u + (2N-4)Ru = 0$, we have

$$(N-2)\big(1+|x|^2\big)Ru + \big(1-|x|^2\big)R^2u = \big(|x|^2-1\big)\sum_{i<j} T_{i,j}^2 u. \tag{3.4}$$

Apply R^{k-1} to both terms, noticing that $R(|x|^2) = 2|x|^2$. One gets

$$\begin{aligned}
&(N-2)\big(1+|x|^2\big)R^k u + (N-2)\sum_{j=1}^{k-1}\binom{k-1}{j} 2^j |x|^2 R^{k-j}u + \big(1-|x|^2\big)R^{k+1}u \\
&\qquad - 2(k-1)|x|^2 R^k u - \sum_{j=2}^{k-1}\binom{k-1}{j} 2^j |x|^2 R^{k-1-j}u \\
&\quad = \sum_{j=0}^{k-1}\binom{k-1}{j} R^j\big(|x|^2-1\big)R^{k-1-j}\left(\sum_{i<j} T_{i,j}^2 u\right).
\end{aligned} \tag{3.5}$$

We proceed by induction on k. Fix a test function $\phi \in C^\infty(S^{N-1})$ and let

$$\varphi(r) = \int_{S^{N-1}} R^k Y u(r\xi)\phi(\xi)\, d\sigma(\xi), \quad 0 < r < 1.$$

Applying Y to formula (3.5) and using the fact that R and Y commute, we find that the induction hypothesis implies that the function

$$g(r) = 2(N-k-1)\varphi(r) + (1-r^2)\big(R\varphi(r) + (2k-N)\varphi(r)\big)$$

has a limit L as $r \to 1$. Solving the differential equation yields

$$\varphi(r) = \frac{(1-r^2)^{N-k-1}}{r^{N-2}} \int_0^r g(t)t^{N-3}(1-t^2)^{k-N}\, dt.$$

If $k \leqslant N-2$, it follows from above that $\varphi(r)$ has limit $L/2(N-k-1)$. If $k = N-1$, then $\varphi(r)$ has a logarithmic growth. □

For an analogous result for $\mathcal{M}$-harmonic functions on the unit ball B^N in $\mathbb{C}^N$ see [5].

4. The Dirichlet problem for hyperharmonic functions

Let $\phi \in C(S^{N-1})$. In this section we look in detail at the solvability of the Dirichlet problem: $\Delta_h u = 0$ in B^N and $u = \phi$ on S^{N-1}. More precisely we prove the following theorem.

Theorem 4.1. *Let $\phi = \sum_{k=0}^{\infty} \phi_k$ be the spherical harmonic expansion of $\phi \in C(S^{N-1})$. Then the Dirichlet problem has a unique solution. It is given by*

$$u(x) = \int_{S^{N-1}} P_h(x,\eta)\phi(\eta)\,d\sigma(\eta), \tag{4.1}$$

where

$$P_h(x,\eta) = \left(\frac{1-|x|^2}{|x-\eta|^2}\right)^{N-1}, \quad x \in B^N,\ \eta \in S^{N-1},$$

or, alternatively, by

$$u(x) = u(ry) = \sum_{k=0}^{\infty} \frac{F_k(r)}{F_k(1)} r^k \phi_k(y), \quad 0 \leqslant r < 1,\ y \in S^{N-1}. \tag{4.2}$$

Proof. For the statement (4.1) see [2]. Let u be a solution of the Dirichlet problem. By Theorem 3.1,

$$u(x) = u(ry) = \sum_{k=0}^{\infty} F(k, 1-N/2, k+N/2; r^2) r^k f_k(y).$$

The proof of Theorem 3.1 given in [7] shows that

$$F_k(r) r^k f_k(y) = \int_{S^{N-1}} u(r\eta) Z_k(y,\eta)\,d\sigma(\eta), \quad k \geqslant 0.$$

Here, Z_k are the zonal harmonics (see [4]). Letting $r \to 1$ we see that $F_k(1) f_k(y) = \phi_k(y)$. This gives (4.2). □

We note that the uniqueness of the solution for the Dirichlet problem shows that for $P_h(x,\eta)$ we have

$$P_h(x,\eta) = \sum_{k=0}^{\infty} \frac{F_k(r)}{F_k(1)} r^k Z_k(y,\eta), \quad x = ry,\ y \in S^{N-1},\ \eta \in S^{N-1}.$$

5. Hardy spaces of hyperharmonic functions

A function $u \in h(B^N)$ is said to belong to the Hardy space $h^p(B^N)$, $0 < p < \infty$, if $M_\alpha u \in L^p(S^{N-1})$, for some (any) $\alpha > 1$. Here, as usual $M_\alpha u$ denotes the non-tangential maximal function defined on S^{N-1} by

$$M_\alpha u(\xi) = \sup\{|u(x)|:\ x \in \Gamma_\alpha(\xi)\},$$

where $\Gamma_\alpha(\xi)$ denotes the non-tangential approach region

$$\Gamma_\alpha(\xi) = \{x \in B^N:\ |x-\xi| < \alpha(1-|x|)\}, \quad \alpha > 1.$$

A function u on B^N is said to have an admissible limit L at $\xi \in S^{N-1}$ if $\lim_{\Gamma_\alpha \ni x \to \xi} = L$.

In this section we show that any function in $h^p(B^N)$ has a distribution value on S^{N-1}.

Proposition 5.1. *Let $u \in h(B^N)$. Then*

(i) *$u = P_h[f]$ for some $f \in L^p(S^{N-1})$, $1 < p < \infty$, if and only if*

$$\sup_{0<r<1} \int_{S^{N-1}} |u(r\xi)|^p \, d\sigma(\xi) < \infty. \tag{5.1}$$

In this case, u has admissible limit f a.e. and $u \in h^p(B^N)$.

(ii) *$u = P_h[\mu]$ for some measure μ if and only if*

$$\sup_{0<r<1} \int_{S^{N-1}} |u(r\xi)| \, d\sigma(\xi) < \infty. \tag{5.2}$$

In this case, u has admissible limit $d\mu/d\sigma$ a.e. Moreover, if $u \in h^1(B^N)$, then $d\mu$ is absolutely continuous.

Proof. If $u = P_h[f]$, obviously, by Hölder's inequality

$$|u(x)|^p \leqslant \int_{S^{N-1}} P_h(x,\xi)|f(\xi)|^p \, d\sigma(\xi).$$

Then (5.1) follows from Theorem 4.1.

Conversely, the fact the L^p-norms are uniformly bounded give the existence of $\varphi \in L^p(S^{N-1})$ and a sequence $r_n \to 1$ such that $u(r_n\xi) \to \varphi(\xi)$ as $n \to \infty$ weakly in $L^p(S^{N-1})$. In particular, for each $x \in B^N$ fixed, by Theorems 3.1 and 4.1,

$$\begin{aligned} P_h[\varphi](x) &= \lim_{n\to\infty} \int_{S^{N-1}} P_h(x,\xi) u(r_n\xi) \, d\sigma(\xi) \\ &= \lim_{n\to\infty} \sum_{k=0}^{\infty} F_k(r_n) r_n^k \frac{F_k(|x|)}{F_k(1)} f_k(x) = u(x). \end{aligned}$$

From the explicit formula for P_h one easily obtains as in the classical case that $M_\alpha u$ is dominated by the Hardy–Littlewood maximal function of f. This implies that $M_\alpha u \in L^p(S^{N-1})$, and the existence of admissible limits is proved in the standard way.

The first part of (ii) is proved similarly. If $u \in h^1(B^N)$, then the convergence of u_r, defined by $u_r(\xi) = u(r\xi)$, $\xi \in S^{N-1}$, is dominated, and hence its weak limit $d\mu$ is absolutely continuous. $\square$

Now we will show that when $p < 1$, and $u \in h^p(B^N)$, then u has a distribution value on S^{N-1}. We will need the technical Lemma 10 from [1].

Lemma 5.2 [1]. *Let $F \in C^2([1/2, 1])$ and $h \in C^1([1/2, 1])$ satisfying $h(1) > -1$. Suppose that*

$$(1-x)F''(x) + h(x)F'(x) = O\big((1-x)^{-A}\big) \quad \text{as } x \to 1.$$

Then:

(i) *If $A > 1$, $F(x) = O((1-x)^{-A+1})$.*
(ii) *If $0 < A < 1$, then there exists $\lim_{x\to 1} F(x)$.*

Theorem 5.3. *Let $u \in h(B^N)$. Assume that for some $p < 1$,*

$$\sup_{0<r<1} \int_{S^{N-1}} \big|u(r\xi)\big|^p \, d\sigma(\xi) < \infty.$$

Then there exists a distribution ϕ satisfying:

(i) *$\lim_{r\to 1} u(r\xi) = \phi(\xi)$ in the sense of distributions.*
(ii) *$u = P_h[\phi]$.*

Proof. Suppose $\varphi \in C^\infty(S^{N-1})$. Define

$$F(r) = \int_{S^{N-1}} u(r\xi)\varphi(\xi)\, d\sigma(\xi).$$

Formula (3.4) gives

$$(1-r^2)\left(r\frac{d}{dr}\right)^2 F(r) + (N-2)(1+r^2)\left(r\frac{d}{dr}\right) F(r) = \int_{S^{N-1}} Xu(r\xi)\varphi(\xi)\, d\sigma(\xi),$$

where $X = (r^2-1)\sum_{i<j} T_{i,j}^2$ is a tangential derivative. Thus, writing $\psi = X^\star \varphi \in C^\infty(S^{N-1})$ with $X^\star$ the adjoint operator we have

$$(1-r^2)F''(r) + \frac{1-r^2+(N-2)(1+r^2)}{r} F'(r) = \int_{S^{N-1}} u(r\xi) r^{-2} \psi(\xi)\, d\sigma(\xi).$$

Iterating the process above and writing

$$L = (1 - r^2)\frac{d^2}{dr^2} + \frac{1 - r^2 + (N-2)(1 + r^2)}{r}\frac{d}{dr}$$

we deduce that for each $k = 1, 2, \ldots$ there exists $\phi_k \in C^\infty(S^{N-1})$ such that

$$(L^k F)(r) = \int\limits_{S^{N-1}} u(r\xi)\phi_k(\xi)\, d\sigma(\xi).$$

Since $\sup_{0<r<1} \int_{S^{N-1}} |u(r\xi)|^p \, d\sigma(\xi) < \infty$, we have

$$\left|u(x)\right| \leqslant \frac{C}{(1 - |x|)^{(N-1)/p}},$$

and consequently $L^k F(r) = O((1 - r)^{-(N-1)/p})$. Applying Lemma 5.2 we find that

$$L^{k-1} F(r) = O\big((1 - r)^{-(N-1)/p+1}\big).$$

Iterating the process we deduce that $\lim_{r\to 1} F(r)$ exists.

Part (ii) follows similarly. □

The proof of Theorem 5.3 shows that a hyperharmonic function u in B^N has a distribution value if and only if $u(x) = O((1 - |x|)^A)$, $A \in \mathbb{R}$. Consequently, as a corollary of Theorems 3.2, 3.4 and 5.3, we have that if $u \in h^p(B^{2n+1})$ (more generally, if u is hyperharmonic in B^{2n+1} and $u(x) = O((1 - |x|)^A)$) and $u \neq \text{const}$, then $\int_{S^{2n}} R^{2n} u(r\xi)\varphi(\xi)\, d\sigma(\xi)$ is $O(\log(1/(1 - r)))$ but is not $o(\log(1/(1 - r)))$ for every $\varphi \in C^\infty(S^{2n})$.

Remark. We note that the results in Sections 4 and 5 are analogous to results on for classical harmonic functions. The Poisson kernel

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^N}, \quad x \in B^N,\ \eta \in S^{n-1},$$

solves the Dirichlet problem for the ordinary Laplacian Δ; it extends distributions on S^{N-1} to classical harmonic functions on B^N in the same way as the "hyperharmonic" Poisson kernel P_h extends distributions on S^{N-1} to hyperharmonic functions on B^N.

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